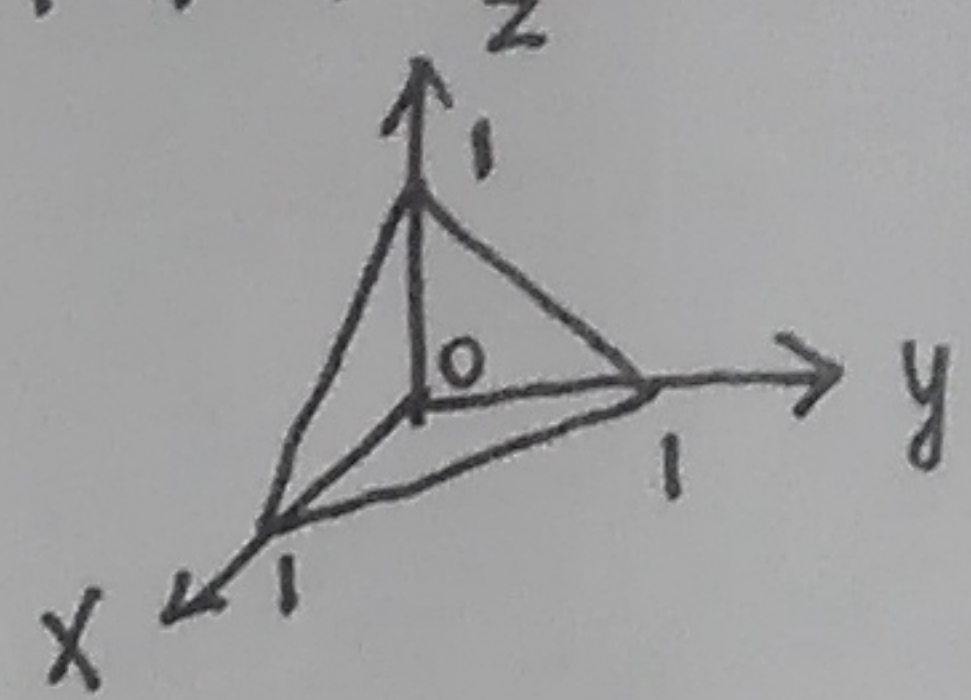


同步练习 P199 二.1

1. 求三重积分 $\iiint_{\Omega} \frac{1}{(1+x+y+z)^3} dv$, 式中 Ω 是由平面 $x+y+z=1$ 与三个坐标面围成的区域.

解: 方法一, 用直角坐标系计算. $\Omega: 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y$

$$\text{于是, } \iiint_{\Omega} \frac{1}{(1+x+y+z)^3} dv = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz$$



$$= \int_0^1 dx \int_0^{1-x} dy \left[-\frac{1}{2}(1+x+y+z)^{-2} \right]_0^{1-x-y} = \int_0^1 dx \int_0^{1-x} \frac{1}{2} \left(\frac{1}{(1+x+y)^2} - \frac{1}{4} \right) dy$$

$$= \int_0^1 dx \left[-\frac{1}{2} \frac{1}{1+x+y} \right]_0^{1-x} - \frac{1}{8}(1-x) = \int_0^1 \left[\frac{1}{2} \frac{1}{1+x} - \frac{1}{4} - \frac{1}{8}(1-x) \right] dx$$

$$= \frac{1}{2} \ln 2 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2} \ln 2 - \frac{5}{16}$$

方法二, 用换元法, 设 $\begin{cases} u=x \\ v=y \\ w=1+x+y+z \end{cases}$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}^{-1} = 1$$

$$1 \leq w \leq 2, \quad 0 \leq v \leq w-1, \quad 0 \leq u \leq w-1-v$$

$$\text{于是, } \iiint_{\Omega} \frac{1}{(1+x+y+z)^3} dv = \int_1^2 \frac{1}{w^3} dw \int_0^{w-1} dv \int_0^{w-1-v} du$$

$$= \int_1^2 \frac{1}{w^3} dw \int_0^{w-1} (w-1-v) dv$$

$$= \int_1^2 \frac{1}{w^3} \cdot \frac{(w-1)^2}{2} dw$$

$$= \frac{1}{2} \int_1^2 \left(\frac{1}{w} - \frac{2}{w^2} + \frac{1}{w^3} \right) dw$$

$$= \frac{1}{2} \left(\ln w + \frac{2}{w} - \frac{1}{2w^2} \right) \Big|_1^2$$

$$= \frac{1}{2} \left(\ln 2 + \frac{2}{2} - \frac{1}{2 \cdot 2^2} \right) - \frac{1}{2} \left(\ln 1 + \frac{2}{1} - \frac{1}{2 \cdot 1^2} \right)$$

$$= \frac{1}{2} \ln 2 + \frac{7}{16} - 1 + \frac{1}{4} = \frac{1}{2} \ln 2 - \frac{5}{16}$$

同步练习 P199 二. 2

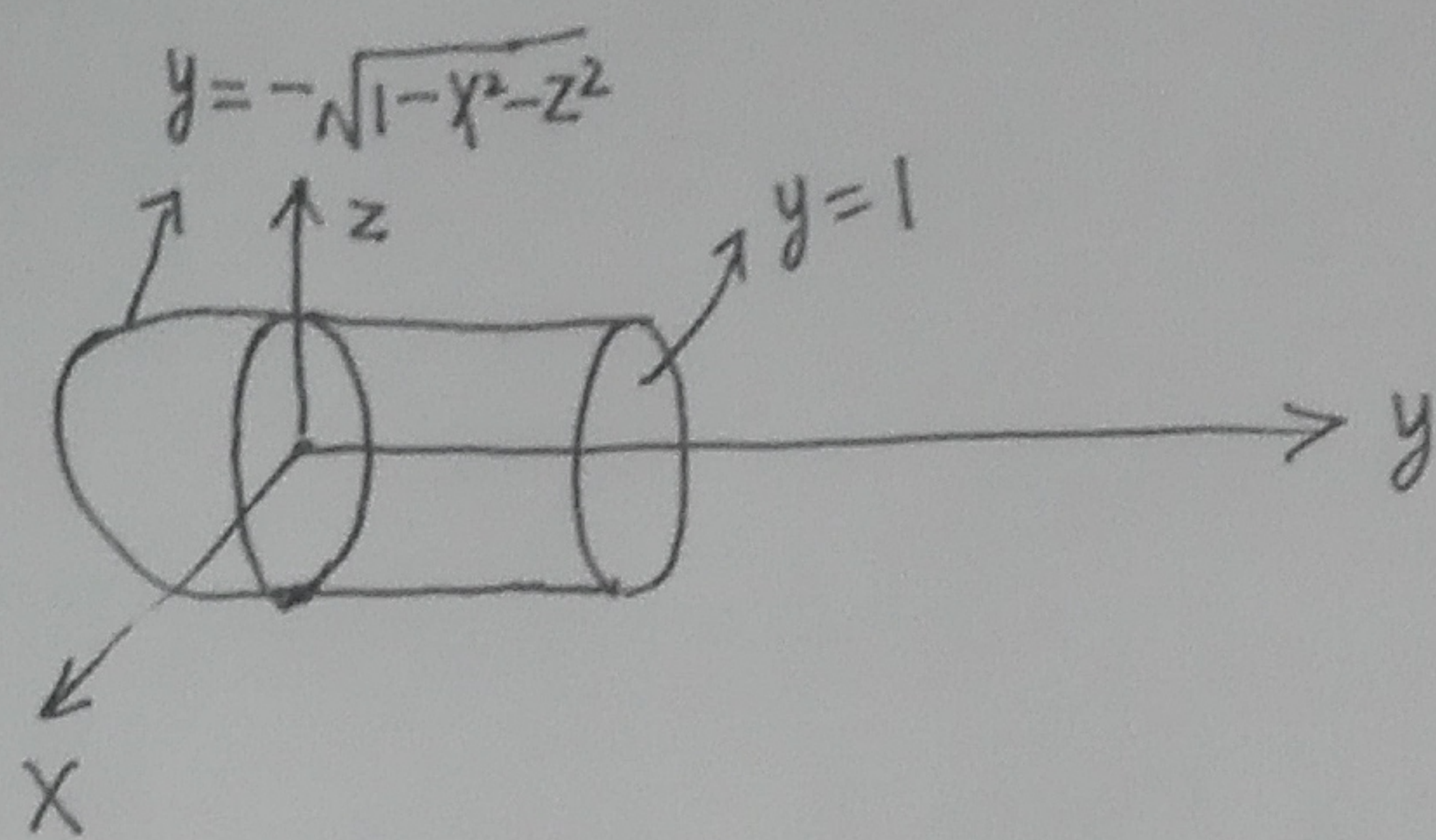
求三重积分 $\iiint_{\Omega} y\sqrt{1-x^2} \, dv$, 式中 Ω 是由曲面 $y = -\sqrt{1-x^2-z^2}$, $x^2+z^2=1$ 及平面 $y=1$ 围成的区域

解: Ω :

$$-1 \leq x \leq 1$$

$$-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}$$

$$-\sqrt{1-x^2-z^2} \leq y \leq 1$$



$$\iiint_{\Omega} y\sqrt{1-x^2} \, dv = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \int_{-\sqrt{1-x^2-z^2}}^1 y\sqrt{1-x^2} \, dy$$

$$= \int_{-1}^1 \sqrt{1-x^2} \, dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2} [1 - (1-x^2-z^2)] \, dz$$

$$= \int_{-1}^1 \frac{\sqrt{1-x^2}}{2} \left(x^2 z + \frac{1}{3} z^3 \right) \Big|_{z=-\sqrt{1-x^2}}^{z=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^1 \frac{\sqrt{1-x^2}}{2} \left(x^2 \cdot 2\sqrt{1-x^2} + \frac{2}{3} (1-x^2)\sqrt{1-x^2} \right) dx$$

$$= \int_{-1}^1 \left[x^2(1-x^2) + \frac{1}{3}(1-x^2)^2 \right] dx$$

$$= \int_{-1}^1 \left(x^2 - x^4 + \frac{1}{3} - \frac{2}{3}x^2 + \frac{1}{3}x^4 \right) dx$$

$$= \int_{-1}^1 \left(\frac{1}{3} + \frac{1}{3}x^2 - \frac{2}{3}x^4 \right) dx$$

$$= \left(\frac{1}{3}x + \frac{1}{9}x^3 - \frac{2}{15}x^5 \right) \Big|_{-1}^1$$

$$= \left(\frac{1}{3} + \frac{1}{9} - \frac{2}{15} \right) - \left(-\frac{1}{3} - \frac{1}{9} + \frac{2}{15} \right) = \frac{14}{45} + \frac{14}{45}$$

$$= \frac{28}{45}$$

同步练习 P200 3. 求三重积分 $\iiint_{\Omega} z^2 dv$, 式中 Ω 由椭球面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 围成

解: (方法一), 用直角坐标计算

$$\Omega: -c \leq z \leq c, -a\sqrt{1-\frac{z^2}{c^2}} \leq x \leq a\sqrt{1-\frac{z^2}{c^2}}, -b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}} \leq y \leq b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}$$

$$\text{于是, } \iiint_{\Omega} z^2 dv = \int_{-c}^c z^2 dz \int_{-a\sqrt{1-\frac{z^2}{c^2}}}^{a\sqrt{1-\frac{z^2}{c^2}}} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}} dy$$

$$= \int_{-c}^c z^2 \pi ab (1 - \frac{z^2}{c^2}) dz = \frac{2}{3} \pi abc^3 - \frac{2}{5} \pi abc^3 = \frac{4}{15} \pi abc^3$$

(方法二) 用换元法, 广义球面坐标 $\begin{cases} x = ar \sin \varphi \cos \theta \\ y = br \sin \varphi \sin \theta \\ z = cr \cos \varphi \end{cases}$

$$|J| = \left| \frac{\partial(x, y, z)}{\partial(r, \varphi, \theta)} \right| = r^2 \sin \varphi \cdot abc$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 化为 $r = 1$ Ω 在 r, φ, θ 坐标系下为单位球体.

$$\text{于是, } \iiint_{\Omega} z^2 dv = \int_0^1 dr \int_0^{\pi} d\varphi \int_0^{2\pi} c^2 r^2 \cos^2 \varphi \cdot r^2 \sin \varphi abc d\theta$$

$$= \int_0^1 r^4 dr \int_0^{\pi} \cos^2 \varphi \sin \varphi d\varphi \int_0^{2\pi} abc^3 d\theta$$

$$= \frac{1}{5} \cdot \frac{2}{3} \cdot 2\pi abc^3 = \frac{4}{15} \pi abc^3$$

(方法三) 用积分的对称性和球面坐标

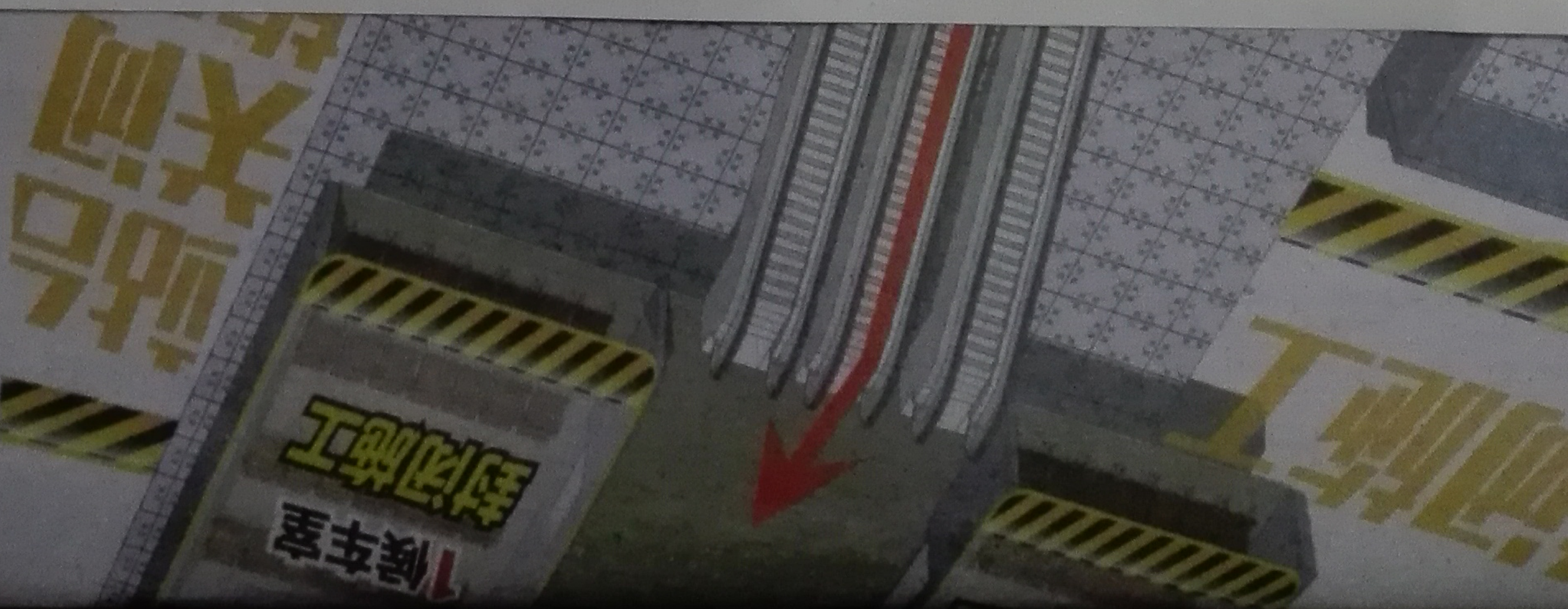
$$\iiint_{\Omega} \frac{z^2}{c^2} dv = \iiint_{\Omega} \frac{y^2}{b^2} dv = \iiint_{\Omega} \frac{x^2}{a^2} dv = \frac{1}{3} \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dv$$

$$= \frac{abc}{3} \int_0^1 dr \int_0^{2\pi} d\theta \int_0^{\pi} r^4 \sin \varphi d\varphi$$

$$= \frac{abc}{3} \cdot \frac{1}{5} \cdot 2\pi \cdot 2 = \frac{4}{15} \pi abc$$

$$\text{于是 } \iiint_{\Omega} \frac{z^2}{c^2} dv \cdot c^2 = \iiint_{\Omega} z^2 dv = \frac{4}{15} \pi abc^3$$

在河南
如家酒店集团正
美军
“无人驾驶”
战舰亮相



同步练习 P200 4

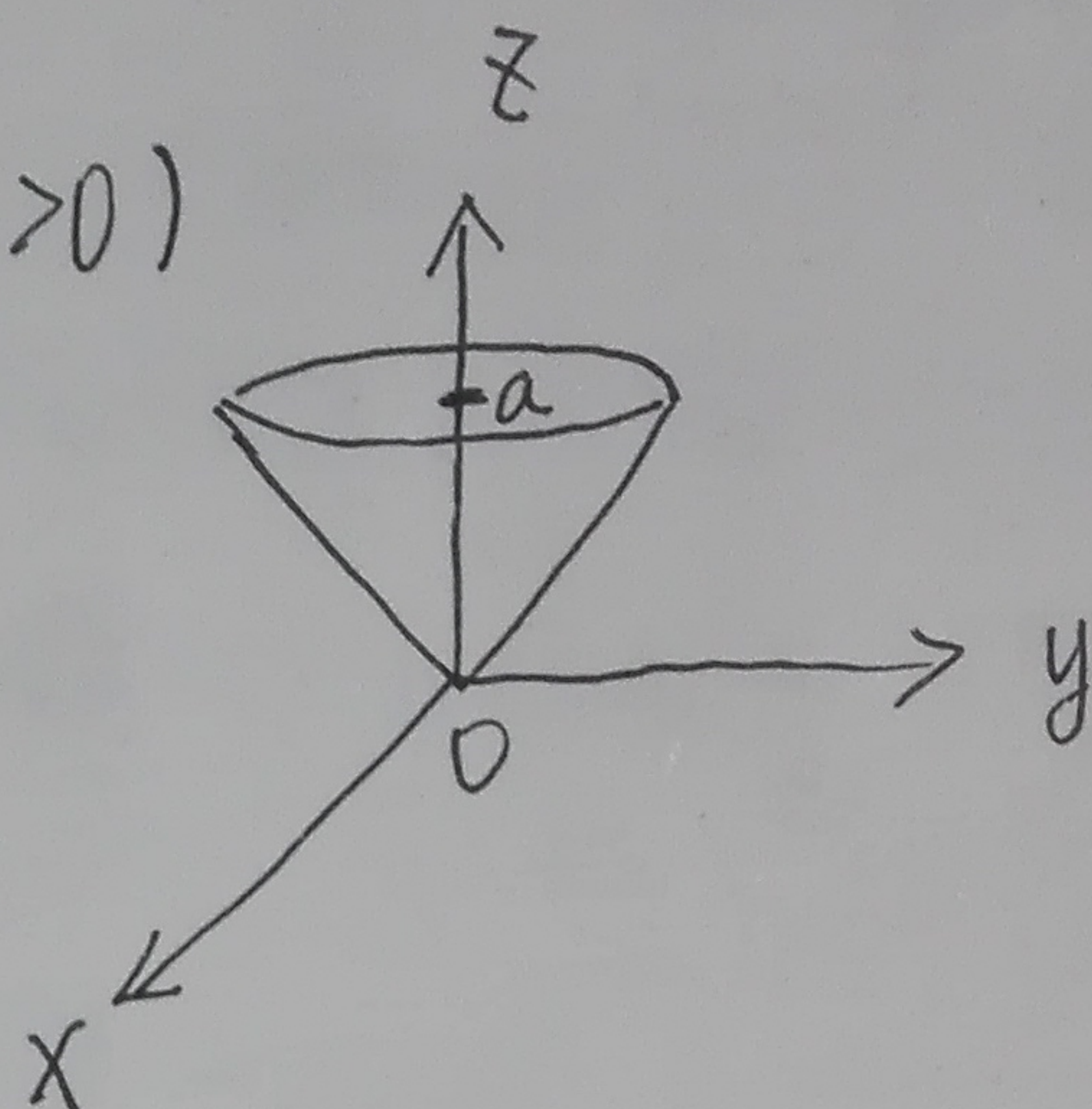
4. 求三重积分 $\iiint_{\Omega} (x^2 + y^2) dv$, 式中 Ω 是由锥面 $x^2 + y^2 = z^2$ 及平面 $z = a$ 围成的区域.

~~解~~: $\Omega = 0 \leq z \leq a$ (这里假设 $a > 0$)

$$0 \leq \rho \leq Z$$

$$0 \leq \theta \leq 2\pi$$

于是，



$$\iiint_V (x^2 + y^2) \, dv = \int_0^a dz \int_0^z \rho^2 \rho \, d\rho \int_0^{2\pi} d\theta$$

$$= \int_0^a dz \int_0^z 2\pi \rho^3 d\rho$$

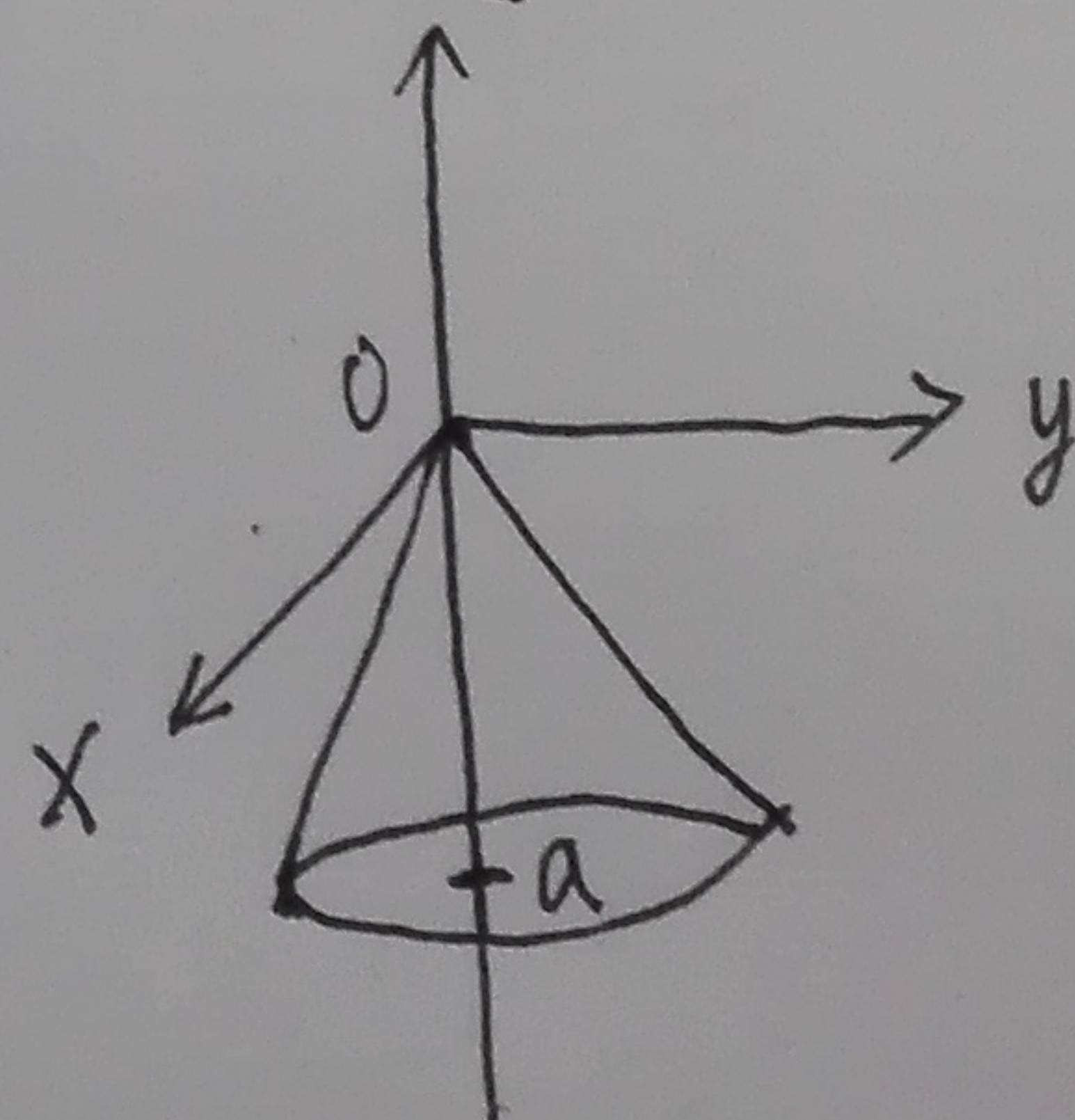
$$= \int_0^a \frac{z}{2} z^4 dz = \frac{\pi}{10} a^5$$

(注, 若 $a < 0$, 积分区域还是锥体, 如下图)

不过这种情况答案为 $\frac{\pi}{10} |a|^5$)

综合考虑, 正确答案为

$$\frac{\pi}{10} |a|^5$$



$$\int_b^a \sin x \cos x \, dx = \frac{1}{2} \sin x \Big|_a^b = \frac{1}{2} \sin b - \frac{1}{2} \sin a$$

同步练习 P200 5

求三重积分 $\iiint_{\Omega} r \cos \theta \, dr d\theta dz$, 其中 $\Omega = \{(\theta, r, z) \mid$

$$0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2, 0 \leq z \leq 1\}.$$

解: $\iiint_{\Omega} r \cos \theta \, dr d\theta dz$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 dr \int_0^1 r \cos \theta \, dz$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 r \cos \theta \, dr$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{2} \cos \theta \, d\theta$$

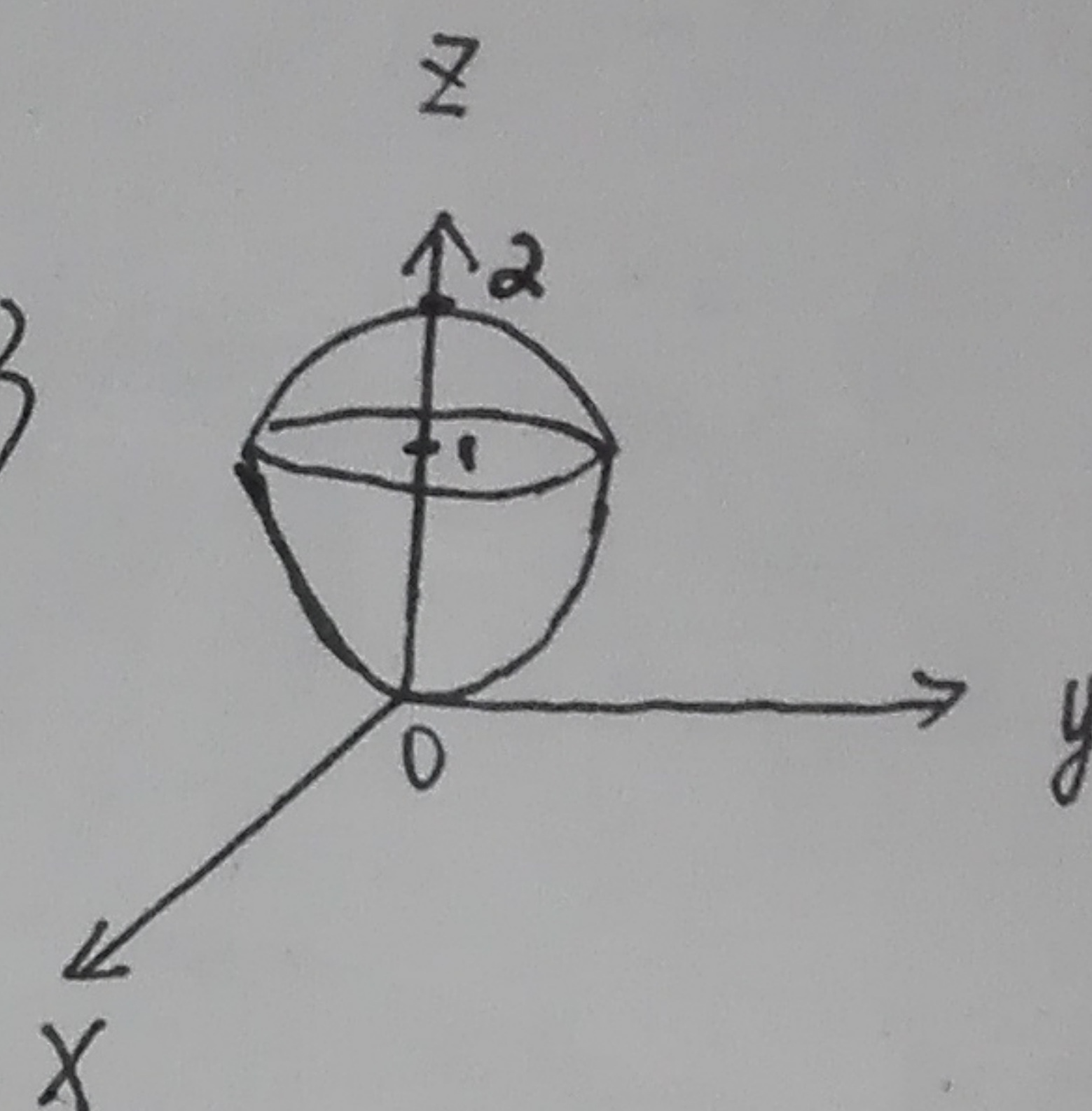
$$= \frac{3}{2}$$

同步练习 P201 6

求三重积分 $\iiint_{\Omega} z dv$, 式中 Ω 是由球面 $x^2 + y^2 + z^2 = 4$ 及旋转抛物面

$z^2 + y^2 = 3z$ 围成的区域

解: 方程组 $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 3z \end{cases}$ 消去 z 得 $x^2 + y^2 = 3z$
 $z = 1$



$\Omega = \Omega_1 + \Omega_2$ 其中

$\Omega_1: 0 \leq z \leq 1, 0 \leq x^2 + y^2 \leq 3z$

$\Omega_2: 1 \leq z \leq 2, 0 \leq x^2 + y^2 \leq 4 - z^2$

于是 $\iiint_{\Omega} z dv = \iiint_{\Omega_1} z dv + \iiint_{\Omega_2} z dv$

$= \int_0^1 z dz \iint_{x^2 + y^2 \leq 3z} dx dy + \int_1^2 \iint_{x^2 + y^2 \leq 4 - z^2} z dx dy$

$= \int_0^1 3z^2 \pi dz + \int_1^2 (4z - z^3) \pi dz$

$= (z^3 \pi) \Big|_0^1 + \pi (2z^2 - \frac{z^4}{4}) \Big|_1^2$

$= \pi + \pi (8 - 4) - \pi (2 - \frac{1}{4})$

$= \frac{13}{4} \pi$

$\int_a^b \sin x \cos x dx = \frac{1}{2} \sin x \Big|_a^b = \frac{1}{2} \sin b - \frac{1}{2} \sin a$

同步练习 P₂₀₁ 7

求三重积分 $\iiint_{\Omega} (y+z) dV$, 式中 Ω 是由锥面 $z = \sqrt{x^2+y^2}$ 与平面 $z=1$ 围成的区域.

解: $\Omega: 0 \leq z \leq 1, x^2+y^2 \leq z^2$

y 关于平面 xOz 对称区域为奇函数.

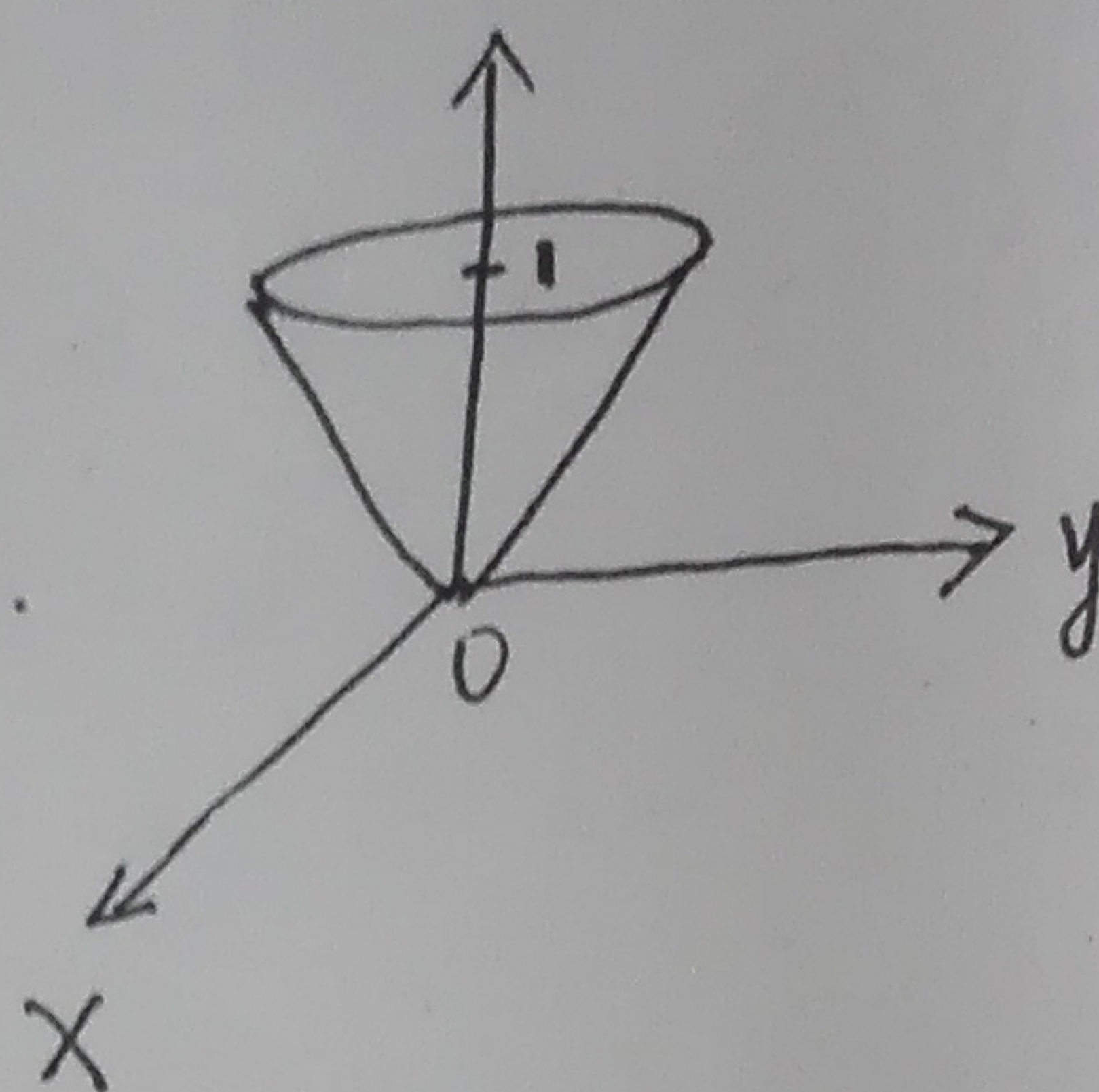
于是, $\iiint_{\Omega} (y+z) dV = \iiint_{\Omega} y dV + \iiint_{\Omega} z dV$

$$= \iiint_{\Omega} z dV$$

$$= \int_0^1 z dz \iint_{x^2+y^2 \leq z^2} dx dy$$

$$= \int_0^1 \pi z^3 dz$$

$$= \frac{\pi}{4}$$



118.02 $\int_0^b \sin x \cos x dx = \frac{1}{2} \sin x /_0^b = \frac{1}{2} \sin b - \frac{1}{2} \sin 0 = \frac{1}{2} \sin b$

同步练习 P201 8

求三重积分 $\iiint_{\Omega} \sqrt{x^2+y^2} dv$, 式中 Ω 是由曲面 $x^2+y^2=2z$ 及平面 $z=2$ 围成的区域.

解: $\Omega: 0 \leq z \leq 2, 0 \leq \rho \leq \sqrt{2z}, 0 \leq \theta \leq 2\pi$

于是, $\iiint_{\Omega} \sqrt{x^2+y^2} dv = \int_0^2 dz \int_0^{\sqrt{2z}} \rho d\rho \int_0^{2\pi} \rho d\theta$

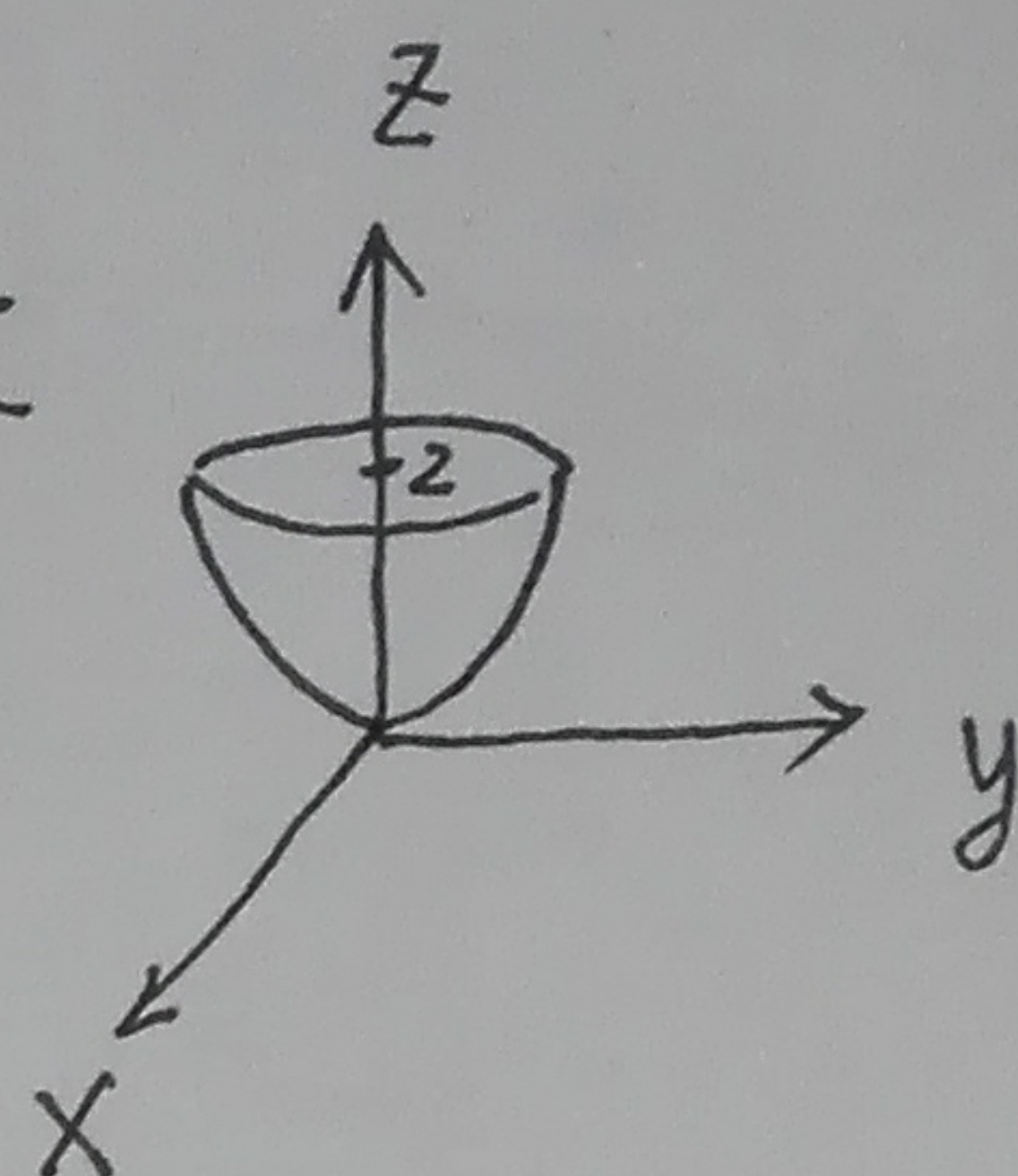
$$= \int_0^2 dz \int_0^{\sqrt{2z}} 2\pi \rho^2 d\rho$$

$$= \int_0^2 \frac{4\sqrt{2}}{3} \pi z^{\frac{3}{2}} dz$$

~~$$= \frac{8\sqrt{2}}{3} \pi$$~~

$$= \pi \frac{8\sqrt{2}}{15} z^{\frac{5}{2}} \Big|_0^2 = \frac{8\sqrt{2}}{15} \cdot 4\sqrt{2} \pi$$

$$= \frac{64}{15} \pi$$



$$\int_0^b \sin x \cos x dx = \frac{1}{2} \sin x \Big|_0^b = \frac{1}{2} \sin b = \frac{1}{2}$$

118.02

同步练习 P202 第9题

9. 求三重积分 $\iiint_{\Omega} |z - x^2 - y^2| dV$, 式中 $\Omega = \{(x, y, z) | x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$

解: $\Omega_1: x^2 + y^2 \leq 1, x^2 + y^2 \leq z \leq 1$

$\Omega_2: x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2$

显然, $\Omega = \Omega_1 + \Omega_2$, 用柱面坐标表示

Ω_1 与 Ω_2 区域:

$\Omega_1: 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq 1, \rho^2 \leq z \leq 1$

$\Omega_2: 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq 1, 0 \leq z \leq \rho^2$

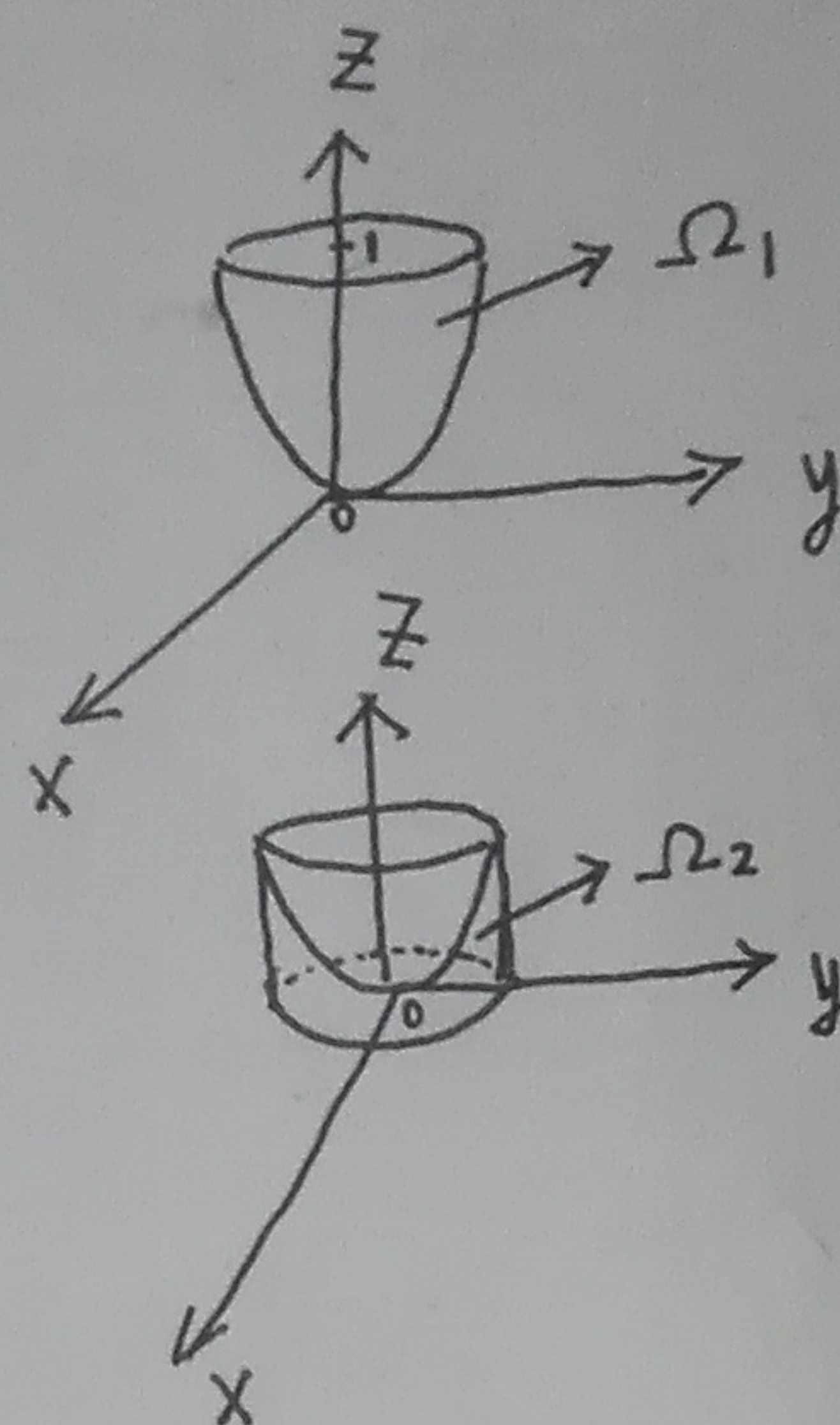
于是, $\iiint_{\Omega} |z - x^2 - y^2| dV = \iiint_{\Omega_1} |z - x^2 - y^2| dV + \iiint_{\Omega_2} |z - x^2 - y^2| dV$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho d\rho \int_{\rho^2}^1 (z - \rho^2) \rho dz + \int_0^{2\pi} d\theta \int_0^1 \rho d\rho \int_0^{\rho^2} (\rho^2 - z) \rho dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(\frac{1}{2} \rho - \rho^3 + \frac{1}{2} \rho^5 \right) d\rho + \int_0^{2\pi} d\theta \int_0^1 \frac{1}{2} \rho^5 d\rho$$

$$= \int_0^{2\pi} \frac{1}{12} d\theta + \int_0^{2\pi} \frac{1}{12} d\theta$$

$$= \frac{\pi}{3}$$



同步练习 P202 10

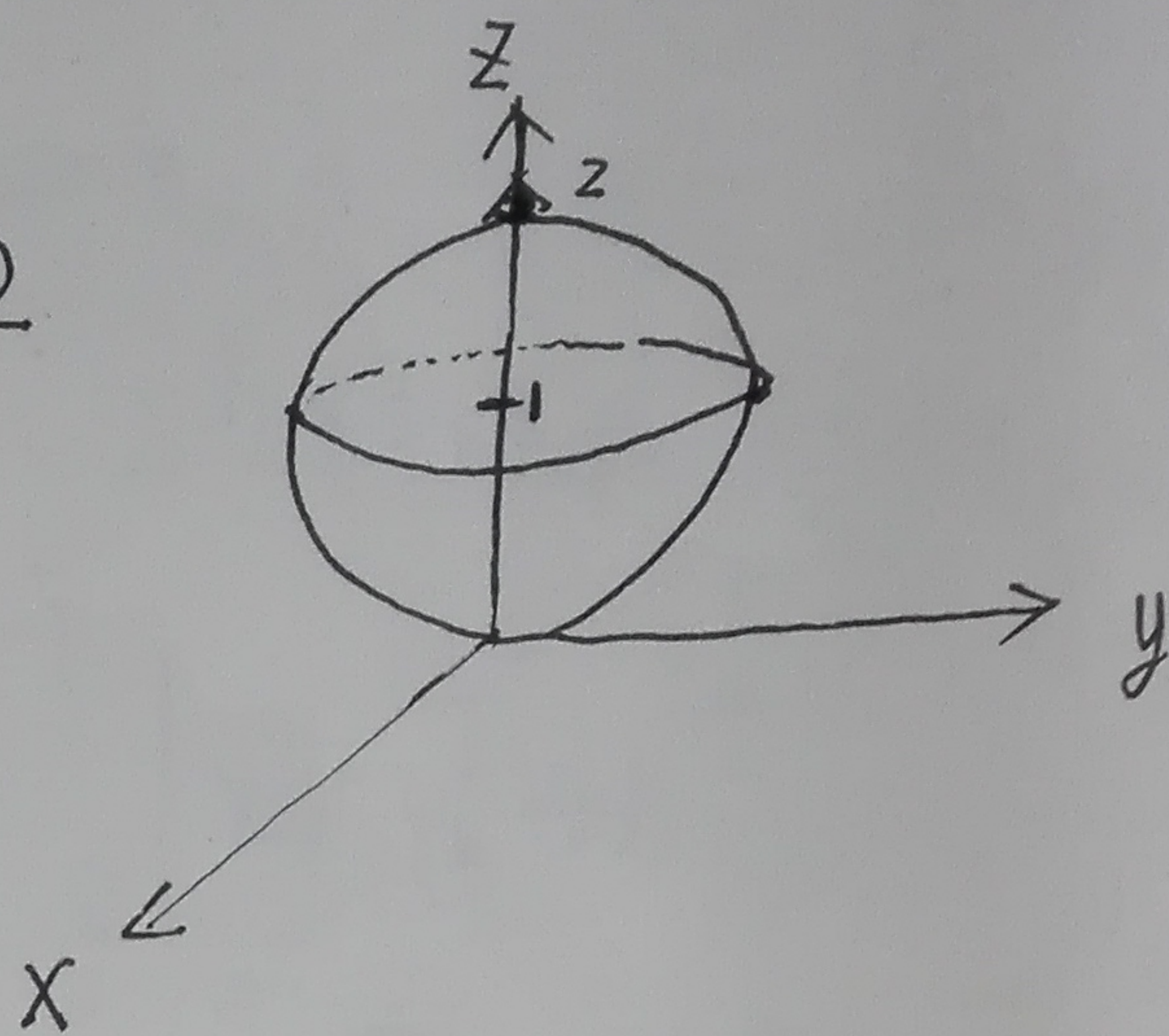
求三重积分 $\iiint_{\Omega} \frac{1}{\sqrt{x^2+y^2+z^2}} dv$, 式中 Ω 是由球面 $x^2+y^2+z^2=2z$ 围成的区域.

解: 用球面坐标表示区域 Ω

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq r \leq 2\cos\varphi$$



于是, $\iiint_{\Omega} \frac{1}{\sqrt{x^2+y^2+z^2}} dv = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \frac{1}{r} \cdot r^2 \sin\varphi dr$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} 2\cos^2\varphi \sin\varphi d\varphi$$

$$= \int_0^{2\pi} \frac{2}{3} d\theta$$

$$= \frac{4}{3} \pi$$

同步练习 P203 12

求由曲面 $z = x^2 + 2y^2$ 与 $z = 6 - 2x^2 - y^2$ 围成的立体体积

解:
$$\begin{cases} z = x^2 + 2y^2 \\ z = 6 - 2x^2 - y^2 \end{cases} \text{ 消去 } z \text{ 得 } x^2 + y^2 = 2$$

立体区域 Ω : $0 \leq x^2 + y^2 \leq 2$, $x^2 + 2y^2 \leq z \leq 6 - 2x^2 - y^2$

$$V = \iiint_{\Omega} dV = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \rho d\rho \int_{\rho^2 + \rho^2 \sin^2 \theta}^{6 - \rho^2 - \rho^2 \cos^2 \theta} dz$$

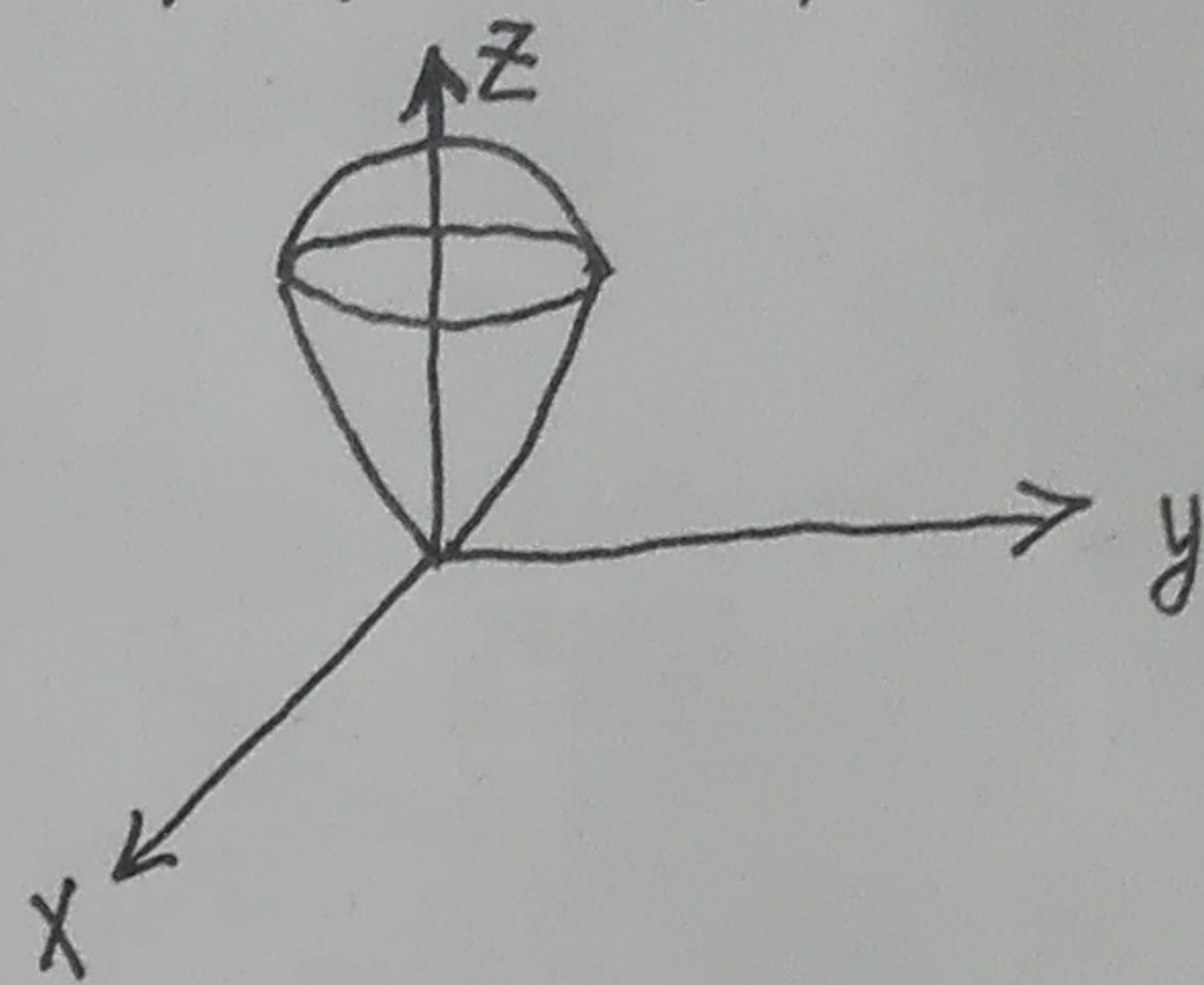
$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \rho(6 - 3\rho^2) d\rho = \int_0^{2\pi} d\theta \cdot \left(3\rho^2 - \frac{3}{4}\rho^4 \right) \Big|_0^{\sqrt{2}}$$

$$= 2\pi \cdot (6 - 3) = 6\pi$$

P202

11. 求三重积分 $\iiint_{\Omega} z \sqrt{x^2 + y^2 + z^2} dV$ 其中 $\Omega = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1, z \geq \sqrt{3(x^2 + y^2)}\}$

解: Ω :
$$\begin{aligned} 0 &\leq \varphi \leq \frac{\pi}{6} \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq r \leq 1 \end{aligned}$$



$$\iiint_{\Omega} z \sqrt{x^2 + y^2 + z^2} dV = \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\pi} d\theta \int_0^1 r^2 \sin\varphi \cdot r \cos\varphi \cdot r dr$$

$$= \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\pi} \sin\varphi \cos\varphi \cdot \frac{1}{5} d\theta$$

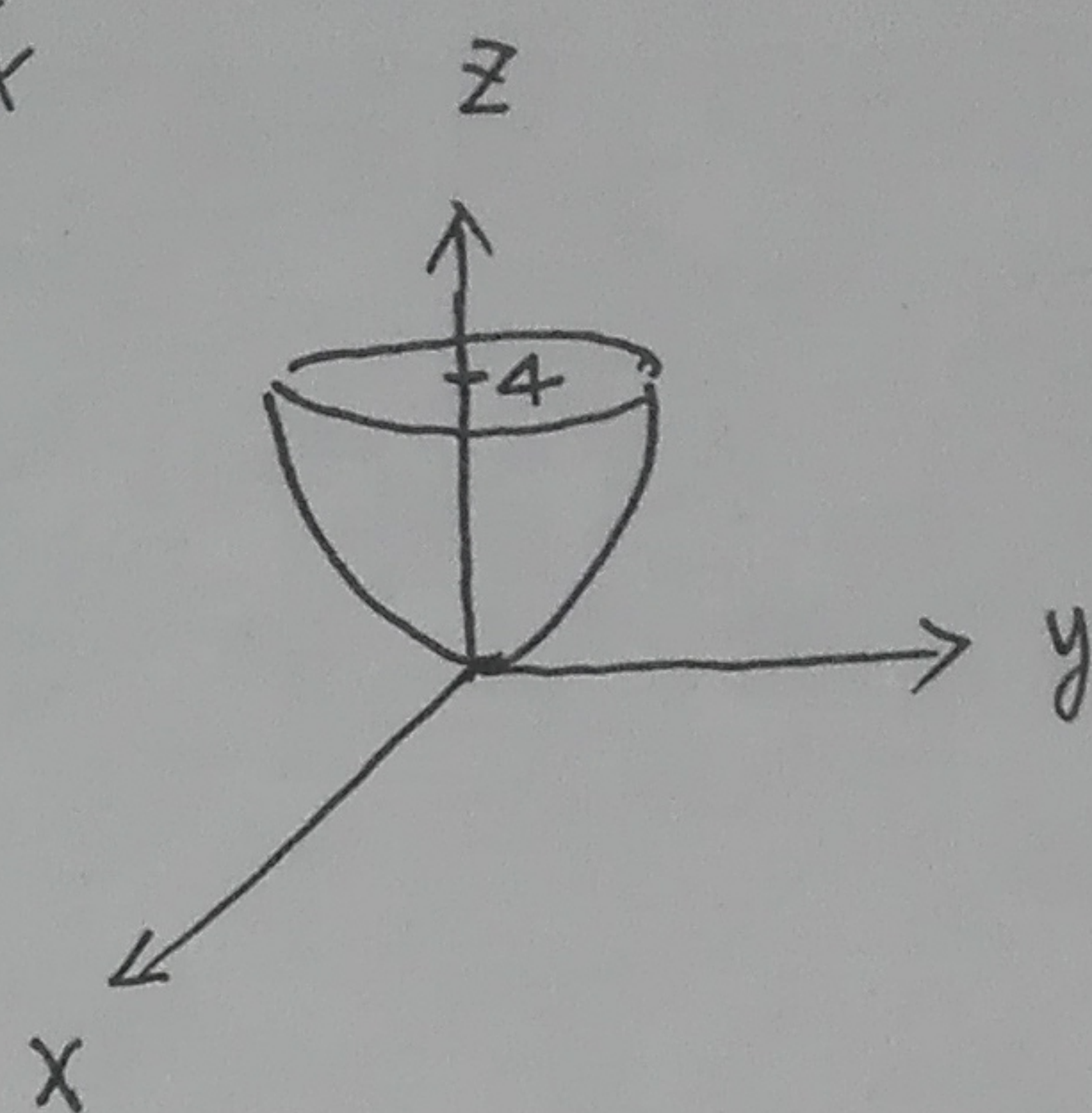
$$= \frac{2\pi}{5} \int_0^{\frac{\pi}{6}} \sin\varphi \cos\varphi d\varphi$$

$$= \frac{\pi}{5} \sin^2\varphi \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{20}$$

13. 求三重积分 $\iiint_{\Omega} (x^2 + y^2 + z^2) dv$, 式中 Ω 是由曲线 $\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$ 绕 z 轴旋转一周形成的曲面与平面 $z = 4$ 围成的区域

解: 用柱面坐标计算

$$\Omega: \begin{aligned} 0 &\leq z \leq 4 \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq \rho \leq \sqrt{2z} \end{aligned}$$



$$\iiint_{\Omega} (x^2 + y^2 + z^2) dv$$

$$= \int_0^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} (\rho^2 + z^2) \rho d\rho$$

$$= \int_0^4 dz \int_0^{2\pi} (z^2 + z^3) d\theta$$

$$= 2\pi \int_0^4 (z^2 + z^3) dz$$

$$= 2\pi \left(\frac{64}{3} + \frac{256}{4} \right)$$

$$= \frac{512}{3} \pi$$

三 证明题 (同步练习 P203)

设 $f(x)$ 是连续函数, 且 $f(0) = 0$, $f'(0) = 2$, 证明:

$$\lim_{t \rightarrow 0} \frac{1}{t^4} \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = 2\pi$$

证明: 三重积分用球面坐标计算,

$$x^2+y^2+z^2 \leq t^2: \quad 0 \leq r \leq |t| \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \varphi \leq \pi$$

$$\text{于是, } \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = \int_0^{|t|} dr \int_0^{2\pi} d\theta \int_0^\pi f(r) \cdot r^2 \sin\varphi d\varphi$$

$$= \int_0^{|t|} dr \int_0^{2\pi} f(r) \cdot r^2 \cdot 2 d\theta$$

$$= \int_0^{|t|} 4\pi f(r) r^2 dr$$

$$\lim_{t \rightarrow 0} \frac{1}{t^4} \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = \lim_{t \rightarrow 0} \frac{\int_0^{|t|} 4\pi f(r) r^2 dr}{t^4}$$

$$\text{设 } |t| = w \quad \lim_{w \rightarrow 0} \frac{\int_0^w 4\pi f(r) \cdot r^2 dr}{w^4} \stackrel{\text{洛必达法则}}{=} \lim_{w \rightarrow 0} \frac{4\pi f(w) \cdot w^2}{4w^3}$$

$$= \lim_{w \rightarrow 0} \pi \frac{f(w)}{w} = \lim_{w \rightarrow 0} \pi \frac{f(w) - f(0)}{w - 0} = \pi \cdot f'(0) = 2\pi$$